Vulnerability of super edge-connected graphs*

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Abstract

A subset F of edges in a connected graph G is a h-extra edge-cut if $G-F$ is disconnected and every component has more than h vertices. The h-extra edge-connectivity $\lambda^{(h)}(G)$ of G is defined as the minimum cardinality over all h-extra edge-cuts of G. A graph G, if $\lambda^{(h)}(G)$ exists, is super-$\lambda^{(h)}$ if every minimum h-extra edge-cut of G isolates at least one connected subgraph of order $h+1$. The persistence $\rho^{(h)}(G)$ of a super-$\lambda^{(h)}$ graph G is the maximum integer m for which $G-F$ is still super-$\lambda^{(h)}$ for any set $F \subseteq E(G)$ with $|F| \leqslant m$. Hong *et al.* [Discrete Appl. Math. 160 (2012), 579-587] showed that $\min\{\lambda^{(1)}(G)-\delta(G)-1, \delta(G)-1\} \leqslant \rho^{(0)}(G) \leqslant \delta(G)-1$, where $\delta(G)$ is the minimum vertex-degree of G. This paper shows that $\min\{\lambda^{(2)}(G)-\xi(G)-1, \delta(G)-1\} \leqslant \rho^{(1)}(G) \leqslant \delta(G)-1$, where $\xi(G)$ is the minimum edge-degree of G. In particular, for a k-regular super-λ' graph G, $\rho^{(1)}(G)=k-1$ if $\lambda^{(2)}(G)$ does not exist or G is super-$\lambda^{(2)}$ and triangle-free, from which the exact values of $\rho^{(1)}(G)$ are determined for some well-known networks.

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1 Introduction

We follow [20] for graph-theoretical terminology and notation not defined here. Let $G = (V, E)$ be a simple connected graph, where $V = V(G)$ is the vertex-set of G and $E = E(G)$ is the edge-set of G. It is well known that when the underlying topology of an interconnection network is modeled by a connected graph $G = (V, E)$, where V is the set of processors and E is the set of communication links in the network, the edge-connectivity $\lambda(G)$ of G is an important measurement for reliability and fault tolerance of the network. In general, the larger $\lambda(G)$ is, the more reliable a network is. Because the connectivity has some shortcomings, Fàbrega and Fiol [5, 6] generalized the concept of the edge-connectivity to the h-extra edge-connectivity for a graph.

Definition 1.1 Let $h \geqslant 0$ be an integer. A subset $F \subseteq E(G)$ is an *h-extra edge-cut* if $G - F$ is disconnected and every component of $G - F$ has more than h vertices. The *h-extra edge-connectivity* of G, denoted by $\lambda^{(h)}(G)$, is defined as the minimum cardinality of an h-extra edge-cut of G.

Clearly, $\lambda^{(0)}(G) = \lambda(G)$ and $\lambda^{(1)}(G) = \lambda'(G)$ for any graph G, the latter is called the restricted edge-connectivity proposed by Esfahanian and Hakimi [4], who proved that for a connected graph G of order at least 4, $\lambda'(G)$ exists if and only if G is not a star.

In general, $\lambda^{(h)}(G)$ does not always exist for $h \geqslant 1$. For example, let $G^*_{n,h}$ $(n \geqslant h)$ be a graph obtained from n copies of a complete graph K_h of order h by adding a new vertex x and linking x to every vertex in each of n copies. Clearly, $G^*_{n,1}$ is a star $K_{1,n}$. It is easy to check that $\lambda^{(h)}(G^*_{n,h})$ does not exists for $h \geqslant 1$.

A graph G is said a *$\lambda^{(h)}$-graph* or to be *$\lambda^{(h)}$-connected* if $\lambda^{(h)}(G)$ exists, and to be *not $\lambda^{(h)}$-connected* otherwise. For a $\lambda^{(h)}$-graph G, an h-extra edge-cut F is a *$\lambda^{(h)}$-cut* if $|F| = \lambda^{(h)}(G)$. It is easy to verify that, for a $\lambda^{(h)}$-graph G,

$$\lambda^{(0)}(G) \leqslant \lambda^{(1)}(G) \leqslant \lambda^{(2)}(G) \leqslant \cdots \leqslant \lambda^{(h-1)}(G) \leqslant \lambda^{(h)}(G). \tag{1.1}$$

For two disjoint subsets X and Y in $V(G)$, use $[X, Y]$ to denote the set of edges between X and Y in G. In particular, $E_G(X) = [X, \overline{X}]$ and let $d_G(X) = |E_G(X)|$, where $\overline{X} = V(G) \setminus X$. For a $\lambda^{(h)}$-graph G, there is certainly a subset $X \subset V(G)$ with $|X| \geqslant h + 1$ such that $E_G(X)$ is a $\lambda^{(h)}$-cut and, both $G[X]$ and $G[\overline{X}]$ are connected. Such an X is called a *$\lambda^{(h)}$-fragment* of G.

For a subset $X \subset V(G)$, use $G[X]$ to denote the subgraph of G induced by X. Let

$$\xi_h(G) = \min\{d_G(X) : \ X \subset V(G),\ |X| = h + 1 \text{ and } G[X] \text{ is connected}\}.$$

Clearly, $\xi_0(G) = \delta(G)$, the minimum vertex-degree of G, and $\xi_1(G) = \xi(G)$, the minimum edge-degree of G defined as $\min\{d_G(x) + d_G(y) - 2 : xy \in E(G)\}$. For a $\lambda^{(h)}$-graph G, Whitney's inequality shows $\lambda^{(0)}(G) \leqslant \xi_0(G)$; Esfahanian and Hakimi [4] showed $\lambda^{(1)}(G) \leqslant \xi_1(G)$; Bonsma *et al.* [1], Meng and Ji [12] showed $\lambda^{(2)}(G) \leqslant \xi_2(G)$. For $h \geqslant 3$, Bonsma *et al.* [1] found that the inequality $\lambda^{(h)}(G) \leqslant \xi_h(G)$ is no longer true in general. The following theorem shows existence of $\lambda^{(h)}(G)$ for any graph G with $\delta(G) \geqslant h$ except for $G^*_{n,h}$.

Theorem 1.2 (Zhang and Yuan [24]) *Let G be a connected graph with order at least $2(\delta+1)$, where $\delta=\delta(G)$. If G is not isomorphic to $G^*_{n,\delta}$, then $\lambda^{(h)}(G)$ exists and*

$$\lambda^{(h)}(G)\leqslant \xi_h(G) \ \text{ for any } h \text{ with } 0\leqslant h\leqslant \delta.$$

A graph G is said to be $\lambda^{(h)}$-*optimal* if $\lambda^{(h)}(G)=\xi_h(G)$. In view of practice in networks, it seems that the larger $\lambda^{(h)}(G)$ is, the more reliable the network is. Thus, investigating $\lambda^{(h)}$-optimal property of networks has attracted considerable research interest (see Xu [19]). A stronger concept than $\lambda^{(h)}$-optimal is super-$\lambda^{(h)}$.

Definition 1.3 A $\lambda^{(h)}$-optimal graph G is *super k-extra edge-connected* (*super-$\lambda^{(h)}$* for short), if every $\lambda^{(h)}$-cut of G isolates at least one connected subgraph of order $h+1$.

By definition, a super-$\lambda^{(h)}$ graph is certainly $\lambda^{(h)}$-optimal, but the converse is not true. For example, a cycle of length $n\,(n\geqslant 2h+4)$ is a $\lambda^{(h)}$-optimal graph and not super-$\lambda^{(h)}$. The following necessary and sufficient condition for a graph to be super-$\lambda^{(h)}$ is simple but very useful.

Lemma 1.4 *A $\lambda^{(h)}$-graph G is super-$\lambda^{(h)}$ if and only if either G is not $\lambda^{(h+1)}$-connected or $\lambda^{(h+1)}(G)>\xi_h(G)$ for any $h\geqslant 0$.*

Faults of some communication lines in a large-scale system are inevitable. However, the presence of faults certainly affects the super connectedness. The following concept is proposed naturally.

Definition 1.5 The *persistence* of a super-$\lambda^{(h)}$ graph G, denoted by $\rho^{(h)}(G)$, is the maximum integer m for which $G-F$ is still super-$\lambda^{(h)}$ for any subset $F\subseteq E(G)$ with $|F|\leqslant m$.

It is clear that the persistence $\rho^{(h)}(G)$ is a measurement for vulnerability of super-$\lambda^{(h)}$ graphs. We can easily obtain an upper bound on $\rho^{(h)}(G)$ as follows.

Theorem 1.6 $\rho^{(h)}(G)\leqslant\delta(G)-1$ *for any super-$\lambda^{(h)}$ graph G.*

Proof. Let G be a super-$\lambda^{(h)}$ graph and F a set of edges incident with some vertex of degree $\delta(G)$. Since $G-F$ is disconnected, $G-F$ is not super-$\lambda^{(h)}$. By the definition of $\rho^{(h)}(G)$, we have $\rho^{(h)}(G)\leqslant\delta(G)-1$. ∎

By Theorem 1.6, we can assume $\delta(G)\geqslant 2$ when we consider $\rho^{(h)}(G)$ for a super-$\lambda^{(h)}$ graph G. In this paper, we only focus on the lower bound on $\rho^{(1)}(G)$ for a super-$\lambda^{(1)}$ graph G. For convenience, we write $\lambda^{(0)}$, $\lambda^{(1)}$, $\lambda^{(2)}$, $\rho^{(0)}$ and $\rho^{(1)}$ for λ, λ', λ'', ρ and ρ', respectively.

Very recently, Hong, Meng and Zhang [7] have showed $\rho(G)\geqslant\min\{\lambda'(G)-\delta(G)-1,\delta(G)-1\}$ for any super-λ and λ'-graph G. In this paper, we establish $\rho'(G)\geqslant\min\{\lambda''(G)-\xi(G)-1,\delta(G)-1\}$, particularly, for a k-regular super-λ' graph G, $\rho'(G)=k-1$ if G is not λ''-connected or super-λ'' and triangle-free. As applications, we determine the exact values of ρ' for some well-known networks.

The left of this paper is organized as follows. In Section 2, we establish the lower bounds on ρ' for general super-λ' graphs. In Section 3, we focus on regular graphs and give some sufficient conditions under which ρ' reaches its upper bound or the difference between upper and lower bounds is at most one. In Section 4, we determine exact values of ρ' for two well-known families of networks.

2 Lower bounds on ρ' for general graphs

In this section, we will establish some lower bounds on ρ' for a general super-λ' graph. The following lemma is useful for the proofs of our results.

Lemma 2.1 (Hellwig and Volkmann [8]) *If G is a λ'-optimal graph, then $\lambda(G) = \delta(G)$.*

Lemma 2.2 *Let G be a λ'-graph and F be any subset of $E(G)$.*

(i) *If G is λ'-optimal and $|F| \leqslant \delta(G) - 1$, then $G - F$ is λ'-connected.*

(ii) *If $G - F$ is λ''-connected, then G is also λ''-connected. Moreover,*

$$\lambda''(G - F) \geqslant \lambda''(G) - |F|. \tag{2.1}$$

Proof. Let G be a λ'-graph of order n and F be any subset of $E(G)$. Clearly, $n \geqslant 4$.

(i) Assume that G is λ'-optimal and $|F| \leqslant \delta(G) - 1$. It is trivial for $\delta(G) = 1$. Assume $\delta(G) \geqslant 2$ below. Since G is λ'-optimal, $\lambda(G) = \delta(G)$ by Lemma 2.1. By $|F| \leqslant \delta(G) - 1$, $G - F$ is connected. If $G - F$ is a star $K_{1,n-1}$, then G has a vertex x with degree $n - 1$. Let $H = G - x$. Then $F = E(H)$ and $\delta(H) \geqslant \delta(G) - 1$. Thus,

$$\delta(G) - 1 \geqslant |F| = |E(H)| \geqslant \frac{1}{2}(n-1)(\delta(G) - 1),$$

which implies $n \leqslant 3$, a contradiction. Thus, $G - F$ is not a star $K_{1,n-1}$, and so is λ'-connected.

(ii) Assume that $G - F$ is λ''-connected, and let X be a λ''-fragment of $G - F$. Clearly, $E_G(X)$ is a 2-extra edge-cut of G, and so G is λ''-connected and $d_G(X) \geqslant \lambda''(G)$. Thus, $\lambda''(G - F) = d_{G-F}(X) \geqslant d_G(X) - |F| \geqslant \lambda''(G) - |F|$. ∎

By Lemma 2.2, we obtain the following result immediately.

Theorem 2.3 *Let G be a super-λ' graph. If G is not λ''-connected, then $\rho'(G) = \delta(G) - 1$.*

Proof. Since G is super-λ', G is λ'-optimal. Let F be any subset of $E(G)$ with $|F| \leqslant \delta(G) - 1$. By Lemma 2.2 (i), $G - F$ is λ'-connected. If G is not λ''-connected, then $G - F$ is also not λ''-connected by Lemma 2.2 (ii). By Lemma 1.4 $G - F$ is super-λ', which implies $\rho'(G) \geqslant \delta(G) - 1$. Combining this with Theorem 1.6, we obtain the conclusion. ∎

By Theorem 2.3, we only need to consider $\rho'(G)$ for a λ''-connected super-λ' graph G. A graph G is said to be *edge-regular* if $d_G(\{x, y\}) = \xi(G)$ for every $xy \in E(G)$, where $d_G(\{x, y\})$ is called the edge-degree of the edge xy in G. Denote by $\eta(G)$ the number of edges with edge-degree $\xi(G)$ in G. For simplicity, we write $\lambda'' = \lambda''(G)$, $\lambda' = \lambda'(G)$, $\rho' = \rho'(G)$, $\xi = \xi(G)$ and $\delta = \delta(G)$ when just one graph G is under discussion.

Theorem 2.4 *Let G be a λ''-connected super-λ' graph. Then*

(i) *$\rho'(G) \geqslant \min\{\lambda'' - \xi - 1, \delta - 1\}$ if $\eta(G) \geqslant \delta$, or*

(ii) *$\rho'(G) \geqslant \min\{\lambda'' - \xi, \delta - 1\}$ if G is edge-regular.*

Proof. Since G is λ''-connected and super-λ', $\lambda'' > \xi$ by Lemma 1.4. If $\delta = 1$, then $\rho' = 0$. Assume $\delta \geqslant 2$ below. Let

$$\begin{aligned} m_1 &= \min\{\lambda'' - \xi - 1, \delta - 1\}, \\ m_2 &= \min\{\lambda'' - \xi, \delta - 1\} \quad \text{and} \quad m = m_1 \quad \text{or} \quad m_2. \end{aligned} \tag{2.2}$$

Since $\lambda'' > \xi$ and $\delta \geqslant 2$, $0 \leqslant m_1 \leqslant \lambda'' - \xi - 1$, $1 \leqslant m_2 \leqslant \lambda'' - \xi$ and $m \leqslant \delta - 1$.

Let F be any subset of $E(G)$ with $|F| = m$ and let $G' = G - F$. Since G is λ'-optimal and $|F| \leqslant \delta - 1$, G' is λ'-connected by Lemma 2.2 (i). To show that $\rho' \geqslant m$, we only need to prove that G' is super-λ'. If G' is not λ''-connected, then G' is super-λ' by Lemma 1.4. Assume now that G' is λ''-connected. It follows from (2.1) and (2.2) that

$$\lambda''(G') \geqslant \lambda''(G) - |F| = \lambda'' - m \geqslant \begin{cases} \xi + 1 & \text{if } m = m_1; \\ \xi & \text{if } m = m_2. \end{cases} \tag{2.3}$$

Since $|F| \leqslant \delta - 1$, if $\eta(G) \geqslant \delta$, G' has at least one edge with edge-degree $\xi(G)$, which implies $\xi(G') \leqslant \xi(G)$. Moreover, if G is edge-regular, then $\eta(G) \geqslant \delta$ and every edge of G is incident with some edge with edge-degree ξ, which implies $\xi(G') < \xi(G)$ if $|F| \geqslant 1$. It follows that

$$\xi(G') \leqslant \begin{cases} \xi(G) & \text{if } \eta(G) \geqslant \delta; \\ \xi(G) - 1 & \text{if } G \text{ is edge-regular and } |F| = m \geqslant 1. \end{cases} \tag{2.4}$$

Combining (2.3) with (2.4), if $m = m_1$ and $\eta(G) \geqslant \delta$ or $m = m_2 \geqslant 1$ and G is edge-regular, we have $\lambda''(G') > \xi(G')$. By Lemma 1.4, G' is super-λ', and so the conclusions (i) and (ii) hold.

The theorem follows. ∎

Remark 2.5 The condition "$\eta(G) \geqslant \delta$" in Theorem 2.4 is necessary. For example, consider the graph G shown in Figure 1, $\eta(G) = 1 < 2 = \delta$. Since $\xi = \lambda = \delta = 2 < 4 = \lambda''$, G is super-λ' by Lemma 1.4. We should have that $\rho'(G) \geqslant \lambda'' - \xi - 1 = \delta - 1 = 1$ by Theorem 2.4, which shows the removal of any edge from G results in a super-λ' graph. However, $\lambda''(G-e) = \xi(G-e) = 4$, and so $G-e$ is not super-λ' by Lemma 1.4, which implies $\rho'(G) = 0$.

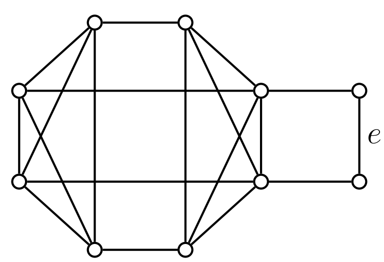


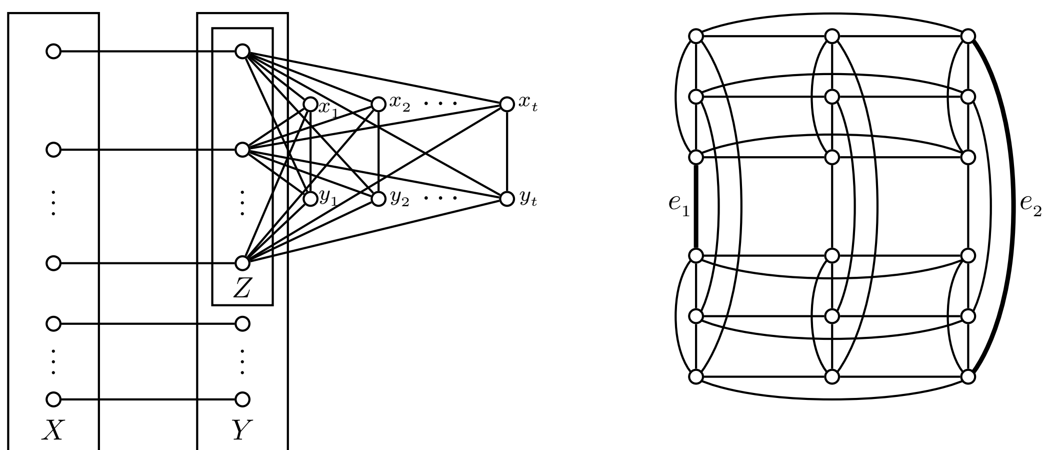
Figure 1: The graph G in Remark 2.5.

The *Cartesian product* of graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertex-set $V(G_1) \times V(G_2)$, two vertices x_1x_2 and y_1y_2, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$, being adjacent in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or $x_2 = y_2$ and $x_1y_1 \in E(G_1)$. The study on λ' for Cartesian products can be found in [9, 10, 13].

Remark 2.6 The graphs G and H shown in Figure 2 can show that the lower bounds on ρ' given in Theorem 2.4 are sharp.

In G, X and Y are two disjoint subsets of $3t-2$ vertices, and Z is a subset of Y with $t-1$ vertices, where $t \geqslant 2$. There is a perfect matching between X and Y and the subgraphs induced by X, Y and $Z \cup \{x_i, y_i\}$ are all complete graphs, for each $i = 1, 2, \ldots, t$. It is easy to check that $\eta(G) = \delta(G) = \lambda(G) = t$, $\lambda'(G) = \xi(G) = 2t-2$, $\lambda''(G) = 3t-2$ and G is super-λ'. By Theorem 2.4, $\rho'(G) \geqslant \lambda''(G) - \xi(G) - 1 = t-1 = \delta(G) - 1$. Combining this fact with Theorem 1.6, we have $\rho'(G) = \delta(G) - 1$. This example shows that the lower bound on ρ' given in Theorem 2.4 (i) is sharp.

For the 5-regular graph $H = K_2 \times K_3 \times K_3$, $\lambda''(H) = 9$ and $\xi(H) = 8$, and so H is super-λ' by Lemma 1.4. On the one hand, by Theorem 2.4, $\rho'(H) \geqslant \lambda''(H) - \xi(H) = 1$. On the other hand, for $F = \{e_1, e_2\}$, $\lambda''(H-F) = 7 = \xi(H-F)$, and so $H-F$ is not super-λ' by Lemma 1.4, which yields $\rho'(H) \leqslant 1$. Hence, $\rho'(H) = \lambda''(H) - \xi(H) = 1$. This example shows that the lower bound on ρ' given in Theorem 2.4 (ii) is sharp.



G $\qquad\qquad$ $H = K_2 \times K_3 \times K_3$

Figure 2: Two graphs G and H in Remark 2.6.

3 Bounds on ρ' for regular graphs

The *girth* of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G. A graph is said to be C_n*-free* if it contains no cycles of length n. In general, C_3-free is said *triangle-free*. To guarantee that G is edge-regular, which is convenient for us to use Theorem 2.4, we consider regular graphs in this section.

Clearly, any k-regular graph contains cycles if $k \geqslant 2$. It is easy to check that C_4 and C_5 are only two 2-regular super-λ' graphs. Obviously, $\rho'(C_4) = \rho'(C_5) = 1$. In the following discussion, we always assume $k \geqslant 3$ when we mention k-regular connected graphs. We first consider 3-regular graphs, such graphs have even order.

Lemma 3.1 *Let G be a 3-regular super-λ' graph of order $2n$. If $n \geqslant 4$, then the grith $g(G) > 4$ and $n \neq 4$.*

Proof. Since G is a 3-regular super-λ' graph of order at least 8, $\lambda'(G) = \xi(G) = 4$, and so $\lambda(G) = 3$ by Lemma 2.1. Moreover, every λ'-cut of G isolates one edge. If G contains a C_3, then let $X = V(C_3)$. If $G - E_G(X)$ isolates a vertex, then $G \cong K_4$, a contradiction with $n \geqslant 4$. Thus, $E_G(X)$ is a 1-extra edge-cut and $\lambda'(G) \leqslant d_G(X) = 3 < 4 = \lambda'(G)$, a contradiction. If G contains a C_4, let $Y = V(C_4)$, then $G - E_G(Y)$ does not isolate a vertex since G contains no triangles, and so $E_G(Y)$ is also a 1-extra edge-cut and $4 = \xi(G) = \lambda'(G) \leqslant d_G(Y) = 4$, which implies that $E_G(Y)$ is λ'-cut of G and does not isolate one edge since $n \geqslant 4$, which means that G is not super-λ', a contradiction. Thus, the girth $g(G) > 4$. Moreover, since any 3-regular graph with girth greater than 4 has at least 10 vertices, we have $n \geqslant 5$. ∎

Theorem 3.2 *Let G be a 3-regular super-λ' graph of order $2n$. If $n = 2$ or 3, then $\rho'(G) = 2$. If $n \geqslant 5$, then $\rho'(G) = 1$.*

Proof. The complete graph K_4 and the complete bipartite graph $K_{3,3}$ are the unique 3-regular super-λ' graphs of order 4 and 6, respectively. It is easy to check that $\rho'(K_4) = \rho'(K_{3,3}) = 2$.

Next, assume $n \geqslant 4$. Then $g(G) > 4$ and $n \geqslant 5$ by Lemma 3.1. Since G is 3-regular super-λ', $\lambda'(G) = \xi(G) = 4$ and every λ'-cut isolates at least one edge. Since $g(G) \geqslant 5$, G is not isomorphic to $G^*_{n,2}$. By Theorem 1.2, G is λ''-connected. By Lemma 1.4 and Theorem 2.4 (ii), $\rho'(G) \geqslant \lambda'' - \xi \geqslant 1$. To prove $\rho'(G) \leqslant 1$, we only need to show that there exists a subset $F \subset E(G)$ with $|F| = 2$ such that $G - F$ is not super-λ'.

Let $P = (u, v, w)$ be a path of length two in G. Since $g(G) > 4$, u and w have only common neighbor v. Let $\{u_1, u_2, v\}$ and $\{w_1, w_2, v\}$ are the sets of neighbors of u and w, respectively. Then either $u_1w_1 \notin E(G)$ or $u_1w_2 \notin E(G)$ since $g(G) > 4$. Assume $u_1w_1 \notin E(G)$ and let $F = \{uu_1, ww_1\}$. Then $\xi(G - F) = 3$. Set $X = V(P)$. Then $d_{G-F}(X) = 3 = \xi(G - F)$. Moreover, it is easy to see that $G[\overline{X}]$ is connected. Thus, X is a 2-extra edge-cut of $G - F$, and so $\lambda''(G - F) \leqslant d_{G-F}(X) = \xi(G - F)$. By Lemma 1.4, $G - F$ is not super-λ', which yields $\rho'(G) \leqslant 1$. Hence, $\rho'(G) = 1$, and so the theorem follows. ∎

The well-known Peterson graph G is a 3-regular super-λ' graph with girth $g(G) = 5$. By Theorem 3.2, $\rho'(G) = 1$.

In general, it is quite difficult to determine the exact value of $\rho'(G)$ of a k-regular super-λ' graph G for $k \geqslant 4$. By Theorem 2.3 for a k-regular super-λ' graph G, if G is not λ''-connected, then $\rho'(G) = k - 1$. Thus, we only need to consider k-regular λ''-graphs. For such a graph G, we can establish some bounds on $\rho'(G)$ in terms of k.

Lemma 3.3 *Let G be k-regular λ''-optimal graph and $k \geqslant 4$. Then G is super-λ' if and only if $g(G) \geqslant 4$ or $k \geqslant 5$.*

Proof. Let G be a k-regular λ''-optimal graph and $k \geqslant 4$. Then $\lambda''(G) = \xi_2(G) = 3k - 4 > 2k - 2 = \xi(G)$ if and only if $g(G) \geqslant 4$, and $\lambda''(G) = \xi_2(G) \geqslant 3k - 6 > 2k - 2 = \xi(G)$ if and only if $k \geqslant 5$. Either of two cases shows that G is super-λ' by Lemma 1.4. ∎

Theorem 3.4 *Let G be a k-regular λ''-optimal graph and $k \geqslant 4$. If $g(G) \geqslant 4$, then*

$$k - 2 \leqslant \rho'(G) \leqslant k - 1.$$

Proof. Since G is λ''-optimal, G is super-λ' by Lemma 3.3. Since G is k-regular and $g(G) \geqslant 4$, $\lambda'' = 3k-4$ and $\xi = 2k-2$. By Theorem 2.4 (ii), $\rho'(G) \geqslant \lambda'' - \xi = k-2$. By Theorem 1.6, $\rho'(G) \leqslant k-1$. ∎

Remark 3.5 The lower bound on ρ' given in Theorem 3.4 is sharp. For example, the 4-dimensional cube Q_4 (see Figure 3) is a 4-regular graph with girth $g = 4$ and $\lambda''(Q_4) = \xi_2(Q_4) = 8$. On the one hand, $\rho'(Q_4) \geqslant 2$ by Theorem 3.4. On the other hand, let X be the subset of vertices of Q_4 whose first coordinates are 0 and $F = \{(0001, 1001), (0010, 1010), (0100, 1100)\}$ (shown by red edges in Figure 3). Since $\lambda''(Q_4 - F) \leqslant d_{Q_4-F}(X) = 5 = \xi(Q_4 - F)$, $Q_4 - F$ is not super-λ' by Lemma 1.4, which implies $\rho'(Q_4) \leqslant 2$. Hence, $\rho'(Q_4) = 2$.

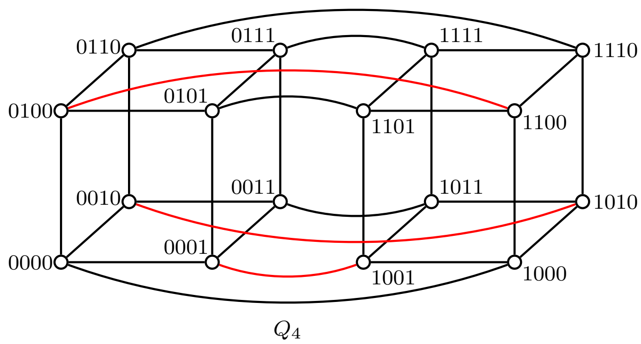


Figure 3: The hypercube Q_4

For a k-regular λ''-optimal graph with $g(G) = 3$, we can establish an upper bound on ρ' under some conditions. To prove our result, we need the following lemma.

Lemma 3.6 (Hong *et al.* [7]) *Let G be an m-connected graph. Then for any subset $X \subset V(G)$ with $|X| \geqslant m$ and $|\overline{X}| \geqslant m$, there are at least m independent edges in $E_G(X)$.*

Theorem 3.7 *Let G be a k-regular λ''-optimal graph with $g(G) = 3$ and $k \geqslant 5$. If G is $(k-2)$-connected and not super-λ'', then*

$$k - 4 \leqslant \rho'(G) \leqslant k - 3, \tag{3.1}$$

and the bounds are best possible.

Proof. Since G is k-regular λ''-optimal, G is super-λ' by Lemma 3.3, $\lambda'' = 3k-6$ and $\xi = 2k-2$. By Theorem 2.4 (ii), $\rho'(G) \geqslant \lambda'' - \xi = k-4$. Thus, we only need to prove $\rho'(G) \leqslant k-3$.

Since G is not super-λ'', there exists a λ''-fragment X of G such that $|\overline{X}| \geqslant |X| \geqslant 4$. Let $|X| = t$. If $t < k-2$, then $k > 6$. For any $x \in X$, since $d_{G[X]}(x) \leqslant t-1$, $|[\{x\}, \overline{X}]| = d_G(x) - d_{G[X]}(x) \geqslant k - t + 1$, and so

$$3k - 6 = \lambda'' = d_G(X) = \sum_{x \in X} |[\{x\}, \overline{X}]| \geqslant t(k - t + 1). \tag{3.2}$$

Since the function $f(t) = t(k - t + 1)$ is convex in the integer interval $[3, k - 2]$ and reaches the minimum value at two end-points of the interval. It follows that

$$f(t) > f(k-2) = f(3) = 3k - 6 \ \text{ for } k > 6. \tag{3.3}$$

Comparing (3.3) with (3.2), we obtain a contradiction. Thus, $t \geqslant k-2$. By Lemma 3.6, there exists a subset $F \subseteq E_G(X)$ consisting of $k - 2$ independent edges. If $G - F$ is not λ''-connected, then $G - F$ is not super-λ' by Lemma 1.4. Assume that $G - F$ is λ''-connected. Then $E_{G-F}(X)$ is a 2-extra edge-cut of $G - F$. Since

$$\lambda''(G - F) \leqslant d_{G-F}(X) = d_G(X) - |F| = 2k - 4 = \xi - 2 \leqslant \xi(G - F),$$

$G - F$ is not super-λ' by Lemma 1.4. Hence $\rho'(G) \leqslant k - 3$.

To show these bounds are best possible, we consider the graph $H = K_2 \times K_3 \times K_3$ and $G = K_4 \times K_4$. For the graph H, it is 5-regular λ''-optimal, and $\rho'(H) = 1$ (see Remark 2.6), which shows that the lower bound given in (3.1) is sharp when $k = 5$. For the graph G, it is 6-regular λ''-optimal but not super-λ''. For any subset $F \subset E(G)$ with $|F| = 3$, $G - F$ is certainly λ''-connected and $\lambda''(G - F) \geqslant \lambda'' - |F| = 12 - 3 = 9 > 8 \geqslant \xi(G - F)$. By Lemma 1.4, $G - F$ is super-λ', which yields $\rho'(G) \geqslant 3$. Hence, $\rho'(G) = 3$, which shows that the upper bound given in (3.1) is sharp.

The theorem follows. ∎

For a k-regular super-λ'' graph, the lower bound on ρ' can be improved a little, which is stated as the following theorem.

Theorem 3.8 *Let G be a k-regular super-λ'' graph and $k \geqslant 4$. Then*

(i) $\rho'(G) = k - 1$ *if* $k \geqslant 4$ *when* $g(G) \geqslant 4$*;*

(ii) $\rho'(G) \geqslant k - 3$ *if* $k \geqslant 6$ *and* $\rho'(G) = 2$ *if* $k = 5$ *when* $g(G) = 3$*.*

Proof. Since G is super-λ'', G is λ''-optimal and $\lambda'' \leqslant 3k - 4$. If $g(G) \geqslant 4$ or $k \geqslant 5$, then G is super-λ' by Lemma 3.3. Let F be any subset of $E(G)$ with $|F| = \lambda'' - \xi + 1$ and $G' = G - F$. Since $|F| = \lambda'' - \xi + 1 \leqslant k - 1$, G' is λ'-connected by Lemma 2.2 (i). We first prove that

$$\rho'(G) \geqslant \lambda'' - \xi + 1 \ \text{ if } g(G) \geqslant 4 \text{ or } k \geqslant 5. \tag{3.4}$$

To the end, we need to prove that G' is super-λ'. By Lemma 1.4, we only need to prove that

$$\lambda''(G') > \xi(G') \ \text{ if } G' \text{ is } \lambda''\text{-connected}. \tag{3.5}$$

Let X be any λ''-fragment of G'. Since $d_{G'}(\overline{X}) = d_{G'}(X) = \lambda''(G')$, we can assume $|X| \leqslant |\overline{X}|$. Since X is a 2-extra edge-cut of G, $d_G(X) \geqslant \lambda''(G) = \lambda''$, and so

$$\lambda''(G') = d_{G'}(X) \geqslant d_G(X) - |F| \geqslant \lambda'' - |F| = \xi - 1. \tag{3.6}$$

On the other hand, since G is edge-regular, we have

$$\xi(G') \leqslant \xi(G) - 1 = \xi - 1. \tag{3.7}$$

Combing (3.6) with (3.7), in order to prove (3.5), we only need to show that at least one of the inequalities (3.6) and (3.7) is strict.

If $F \not\subseteq E_G(X)$, $d_{G'}(X) > d_G(X) - |F|$, and so the first inequality in (3.6) is strict. Assume $F \subset E_G(X)$ below. If $|X| \geqslant 4$, then $E_G(X)$ is not a λ''-cut since G is super-λ'', which implies $d_G(X) > \lambda''$, and so the second inequality in (3.6) is strict.

Now, consider $|X| = 3$ and we have the following two subcases.

If $g(G) \geqslant 4$, then $\lambda'' = 3k - 4$, and so $|F| = \lambda'' - \xi + 1 = k - 1 \geqslant 3$. Since $F \subset E_G(X)$, there exists one edge in $G[X]$ which is adjacent to at least two edges of F, which implies $\xi(G') \leqslant \xi - 2 < \xi - 1$, that is, the inequality (3.7) is strict.

If $g(G) = 3$, then $\lambda'' = 3k - 6$. If $G[X]$ is not a triangle, $d_G(X) = 3k - 4 > 3k - 6 = \lambda''$, and so the second inequality in (3.6) is strict. If $G[X]$ is a triangle, since $|F| = \lambda'' - \xi + 1 = k - 3 \geqslant 2$ and $F \subset E_G(X)$, then there exists one edge in $G[X]$ which is adjacent to at least two edges of F, which implies $\xi(G') \leqslant \xi - 2 < \xi - 1$, that is, the inequality (3.7) holds strictly.

Thus, the inequality (3.5) holds, and so the inequality (3.4) follows. We now prove the remaining parts of our conclusions.

(i) When $g(G) \geqslant 4$, $\lambda'' = 3k - 4$. By Theorem 1.6 and (3.4), $k - 1 \geqslant \rho'(G) \geqslant \lambda'' - \xi + 1 = k - 1$, which implies $\rho'(G) = k - 1$.

(ii) When $g(G) = 3$, $\lambda'' = 3k - 6$. By (3.4), $\rho'(G) \geqslant \lambda'' - \xi + 1 = k - 3$. If $k = 5$, $\rho'(G) \geqslant 2$. Choose a subset $X \subset V(G)$ such that $G[X]$ is a triangle. It is easy to check that $E_G(X)$ is a λ''-cut. Let F be a set of three independent edges of $E_G(X)$. Then $\lambda''(G - F) \leqslant d_{G-F}(X) = 6 = \xi(G - F)$. This fact shows that $G - F$ is not super-λ', which implies $\rho'(G) \leqslant 2$. Thus, $\rho'(G) = 2$.

The theorem follows. ∎

A graph G is *transitive* if for any two given vertices u and v in G, there is an automorphism ϕ of G such that $\phi(u) = v$. A transitive graph is always regular. The studies on extra edge-connected transitive graphs and super extra edge-connected transitive graphs can be found in [11, 16, 22, 23] etc.

Lemma 3.9 (Wang and Li [16]) *Let G be a connected transitive graph of degree $k \geqslant 4$ with girth $g \geqslant 5$. Then G is λ''-optimal and $\lambda''(G) = 3k - 4$.*

Lemma 3.10 (Yang *et al.* [23]) *Let G be a C_4-free transitive graph of degree $k \geqslant 4$. If G is λ''-optimal, then G is super-λ''.*

Combining Theorem 3.8 (i) with Lemma 3.9 and Lemma 3.10, we have the following corollary immediately.

Corollary 3.11 *If G is a connected transitive graph of degree $k \geqslant 4$ with girth $g \geqslant 5$, then $\rho'(G) = k - 1$.*

Remark 3.12 In Corollary 3.11, the condition "$g \geqslant 5$" is necessary. For example, the connected transitive graph Q_4 is λ''-optimal and not super-λ'', and $\rho'(Q_4) = 2$ (see Remark 3.5).

4 ρ' for two families of networks

As applications of Theorem 3.8 (i), in this section, we determine the exact values of $\rho'(G)$ for two families of networks $G(G_0, G_1; M)$ and $G(G_0, G_1, \ldots, G_{m-1}; \mathscr{M})$ subject to some conditions.

The first family of networks $G(G_0, G_1; M)$ is defined as follows. Let G_0 and G_1 be two graphs with the same number of vertices. Then $G(G_0, G_1; M)$ is the graph G with vertex-set $V(G) = V(G_0) \cup V(G_1)$ and edge-set $E(G) = E(G_0) \cup E(G_1) \cup M$, where M is an arbitrary perfect matching between vertices of G_0 and G_1. Thus the hypercube Q_n, the twisted cube TQ_n, the crossed cube CQ_n, the Möbius cube MQ_n and the locally twisted cube LTQ_n all can be viewed as special cases of $G(G_0, G_1; M)$ (see [2]).

The second family of networks $G(G_0, G_1, \ldots, G_{m-1}; \mathscr{M})$ is defined as follows. Let $G_0, G_1, \ldots, G_{m-1}$ be m ($\geqslant 3$) graphs with the same number of vertices. Then $G(G_0, G_1, \ldots, G_{m-1}; \mathscr{M})$ is the graph G with vertex-set $V(G) = V(G_0) \cup V(G_1) \cup \cdots \cup V(G_{m-1})$ and edge-set $E(G) = E(G_0) \cup E(G_1) \cup \cdots \cup E(G_{m-1}) \cup \mathscr{M}$, where $\mathscr{M} = \cup_{i=0}^{m-1} M_{i,i+1(\mathrm{mod}\ m)}$ and $M_{i,i+1(\mathrm{mod}\ m)}$ is an arbitrary perfect matching between $V(G_i)$ and $V(G_{i+1(\mathrm{mod}\ m)})$. Recursive circulant graphs [15] and the undirected toroidal mesh [19] are special cases of this family.

The super edge-connectivity of above two families of networks is studied by Chen *et al.* [2]. Chen and Tan [3] further studied the restricted edge-connectivity of above two families of networks, and $\lambda'(G(G_0, G_1; M))$ is also studied by Xu *et al.* [21]. The 2-extra edge-connectivity of above two families of networks is studied by Wang *et al.* [18]. The vulnerability ρ of super edge-connectivity of the two families of networks is discussed by Wang and Lu [17]. In this section, we will further investigate the vulnerability ρ' of the two families of super-λ' networks without triangles.

Lemma 4.1 (see Example 1.3.1 in Xu [20]) *If G is a triangle-free graph of order n, then $|E(G)| \leqslant \frac{n^2}{4}$.*

We consider the first family of graphs $G = G(G_0, G_1; M)$ for k-regular triangle-free and super-λ graphs G_0 and G_1. Under these hypothesis, G is $(k+1)$-regular and triangle-free. By Theorem 3.2, we can assume $k \geqslant 3$. We attempt to use Theorem 3.8 (i) to determine the exact value of $\rho'(G)$ when G is super-λ''. However, there are some such graphs that are not super-λ''.

Example 4.2 Let G_0 be a k-regular triangle-free and super-λ graph of order n. Then G_0 is λ'-connected, $k \geqslant 3$ and $n \geqslant 6$. $G = G_0 \times K_2$ can be viewed as $G(G_0, G_0; M)$ for some perfect matching M. Assume $n \leqslant 3k - 1$ or $\lambda'(G_0) \leqslant \frac{3k-1}{2}$. If the former happens, then M is a 2-extra edge-cut, and so $\lambda''(G_0 \times K_2) \leqslant |M| = 3k - 1$. However, $G_0 \times K_2$ is not super-λ'' since $n \geqslant 6$. If the latter happens, let $X_0 \subset V(G_0)$ such that $E_{G_0}(X_0)$ is a λ'-cut of G_0, then $G[X_0] \times K_2 \subset G$. Let $Y = V(G[X_0] \times K_2)$. Since $E_G(Y)$ is a 2-extra edge-cut, $\lambda''(G_0 \times K_2) \leqslant |E_G(Y)| = 2\lambda'(G_0) \leqslant 3k - 1$. However, $G_0 \times K_2$ is not super-λ'' since $|Y| \geqslant 4$.

This example shows that the condition "$\min\{n, \lambda'(G_0) + \lambda'(G_1)\} > 3k - 1$" is necessary to guarantee that $G = G(G_0, G_1; M)$ is super-λ''. Thus, we can state our result as follows.

Theorem 4.3 *Let G_i be a triangle-free k-regular and super-λ graph of order n for each $i = 0, 1$. If $\min\{n, \lambda'_0 + \lambda'_1\} > 3k - 1$, then $G = G(G_0, G_1; M)$ is super-λ'' and $\rho'(G) = k$, where $\lambda'_i = \lambda'(G_i)$ for each $i = 0, 1$.*

Proof. Clearly, $k \geqslant 3$. Since G is $(k+1)$-regular and triangle-free, by Theorem 3.8 (i), we only need to prove that G is super-λ''. Since M is a 2-extra edge-cut of G, $\lambda''(G)$ exists. By Theorem 1.2,

$$\lambda''(G) \leqslant \xi_2(G) = 3k - 1. \tag{4.1}$$

Suppose to the contrary that G is not super-λ''. Then there exists a λ''-fragment X of G such that $|\overline{X}| \geqslant |X| \geqslant 4$. Since G is triangle-free, $G[X]$ is also triangle-free, and so $|E(G[X])| \leqslant \frac{|X|^2}{4}$ by Lemma 4.1. It follows that

$$3k - 1 \geqslant d_G(X) = (k+1)|X| - 2|E(G[X])| \geqslant (k+1)|X| - \tfrac{1}{2}|X|^2,$$

that is, $(|X|-3)(|X|-(2k-1))+1 \geqslant 0$, which implies that, since $|X| \geqslant 4$ and $k \geqslant 3$,

$$|X| \geqslant 2k - 1. \tag{4.2}$$

We will deduce a contradiction to (4.1) by proving that

$$\lambda''(G) > 3k - 1. \tag{4.3}$$

To the end, set $V_i = V(G_i)$ and $X_i = X \cap V_i$ for each $i = 0, 1$. There are two cases.

Case 1. Exactly one of X_0 and X_1 is empty.

Without loss of generality, assume $X = X_0$. Then $E_G(X) = E_{G_0}(X) \cup [X, V_1]$. By the definition of G, $|[X, V_1]| = |X|$, and so

$$\lambda''(G) = d_G(X) = d_{G_0}(X) + |X|. \tag{4.4}$$

It is easy to check that $G_0[V_0 \setminus X]$ is connected. Thus, when $2 \leqslant |X| \leqslant n-2$, $E_{G_0}(X)$ is a 1-extra edge-cut of G_0, and so $d_{G_0}(X) \geqslant \lambda'_0$. Since G_0 is super-λ, $\lambda'_0 > \lambda(G_0) = k$, and so

$$d_{G_0}(X) \geqslant \begin{cases} k+1 & \text{if } 2 \leqslant |X| \leqslant n-2, \\ k & \text{if } |X| = n-1, \\ 0 & \text{if } |X| = n. \end{cases} \tag{4.5}$$

Substituting $n > 3k - 1$, (4.2) and (4.5) into (4.4) yields the inequality (4.3).

Case 2. $X_0 \neq \emptyset$ and $X_1 \neq \emptyset$.

Assume that one of $G[X_0]$, $G[X_1]$, $G[V_0 \setminus X_0]$ and $G[V_1 \setminus X_1]$ is not connected. Without loss of generality, assume that $G[X_0]$ has two components H and T. Then $[H, V_0 \setminus X_0] \cup [T, V_0 \setminus X_0] \cup [X_1, V_1 \setminus X_1] \subseteq E_G(X)$, and the first two are edge-cuts of G_0, and the last is an edge-cut of G_1. Since G_i is super-λ, $\lambda(G_i) = k$ for each $i = 0, 1$. Thus,

$$\lambda''(G) = |E_G(X)| \geqslant |[H, V_0 \setminus X_0]| + |[T, V_0 \setminus X_0]| + |[X_1, V_1 \setminus X_1]| \geqslant 3k > 3k - 1,$$

and so (4.3) follows.

Now, we assume that all of $G[X_0]$, $G[X_1]$, $G[V_0 \setminus X_0]$ and $G[V_1 \setminus X_1]$ are connected. Since $|X| \geqslant 4$, $\max\{|X_0|, |X_1|\} \geqslant 2$. We consider the following two subcases.

Subcase 2.1. $|X_0| \geqslant 2$ and $|X_1| \geqslant 2$.

In this case, $E_{G_0}(X_0) \cup E_{G_1}(X_1) \subseteq E_G(X)$. For each $i = 0, 1$, $d_{G_i}(X_i) \geqslant \lambda'_i$ since $E_{G_i}(X_i)$ is a 1-extra edge-cut of G_i. By our hypothesis,

$$\lambda''(G) = d_G(X) \geqslant d_{G_0}(X_0) + d_{G_1}(X_1) \geqslant \lambda'_0 + \lambda'_1 > 3k - 1,$$

and so (4.3) follows.

Subcase 2.2. Exact one of X_0 and X_1 is a single vertex.

Without loss of generality, assume $|X_0| = 1$. Then $|X_1| = |X| - 1 \geqslant 2k - 2$ by (4.2). Clearly,

$$E_G(X) = E_{G_0}(X_0) \cup E_{G_1}(X_1) \cup [X_1, V_0 \setminus X_0],$$

$d_{G_0}(X_0) = k$ and $|[X_1, V_0 \setminus X_0]| = |X| - 2 \geqslant 2k - 3$, and so

$$\lambda''(G) = d_G(X) \geqslant k + d_{G_1}(X_1) + 2k - 3. \tag{4.6}$$

If $2 \leqslant |X_1| \leqslant n - 2$, then X_1 is a 1-extra edge-cut of G_1, and so $d_{G_1}(X_1) \geqslant \lambda'_1 > \lambda(G_1) = k$ since G_1 is super-λ. If $|X_1| = n - 1$, then $E_{G_1}(X_1)$ isolates a vertex, and so $d_{G_1}(X_1) = k$. Thus, we always have $d_{G_1}(X_1) \geqslant k$. Substituting this inequality into (4.6) yields (4.3) since $d_G(X) \geqslant k + k + 2k - 3 = 4k - 3 > 3k - 1$ for $k \geqslant 3$.

Under the hypothesis that G is not super-λ'', we deduce a contradiction to (4.1). Thus, G is super-λ''. By Theorem 3.8 (i), $\rho'(G) = k$, and so the theorem follows. ∎

Lemma 4.4 (Xu *et al.* [21]) *If $G_n \in \{Q_n, TQ_n, CQ_n, MQ_n, LTQ_n\}$, then $\lambda'(G_n) = 2n - 2$ and, thus, G_n is λ'-optimal for $n \geqslant 2$, and is super-λ for $n \geqslant 3$.*

Corollary 4.5 *Let $G_n \in \{Q_n, TQ_n, CQ_n, MQ_n, LTQ_n\}$. If $n \geqslant 5$, then G_n is super-λ'', super-λ' and $\rho'(G_n) = n - 1$.*

Proof. Let $G_n \in \{Q_n, TQ_n, CQ_n, MQ_n, LTQ_n\}$. Then G_n can be viewed as the graph $G(G_{n-1}, G_{n-1}; M)$ corresponding to some perfect matching M. G_{n-1} is an $(n-1)$-regular and triangle-free graph of order 2^{n-1}. By Lemma 4.4, G_{n-1} is super-λ and $\lambda'(G_{n-1}) = 2n - 4$ for $n \geqslant 4$. Thus, $2\lambda'(G_{n-1}) = 4n - 8 > 3(n-1) - 1$ and $2^{n-1} > 3(n-1) - 1$ for $n \geqslant 5$. By Theorem 4.3, G_n is super-λ'' and $\rho'(G_n) = n - 1$ if $n \geqslant 5$. Hence, if $n \geqslant 5$, $\lambda''(G_n) = 3n - 4 > 2n - 2 = \xi(G_n)$ implies G_n is super-λ'. ∎

Remark 4.6 In Corollary 4.5, the condition "$n \geqslant 5$" is necessary. For example, Q_4 is λ''-optimal and not super-λ'', and $\rho'(Q_4) = 2$ (see Remark 3.5).

We now consider the second family of graphs $G(G_0, G_1, \ldots, G_{m-1}; \mathscr{M})$. To guarantee that G is triangle-free, we can assume $m \geqslant 4$. Let $I_m = \{0, 1, \ldots, m-1\}$.

Theorem 4.7 *Let G_i be a k-regular k-edge-connected graph of order n without triangles for each $i \in I_m$. If $k \geqslant 3$, $n > \lceil \frac{3k+2}{2} \rceil$ and $m \geqslant 4$, then $G = G(G_0, \ldots, G_{m-1}; \mathscr{M})$ is super-λ'' and $\rho'(G) = k + 1$.*

Proof. It is easy to check that G is $(k+2)$-regular and triangle-free. By Theorem 1.2, G is λ''-connected and

$$\lambda''(G) \leqslant \xi_2(G) = 3k + 2. \tag{4.7}$$

By Theorem 3.8 (i), we only need to prove that G is super-λ''. Suppose to the contrary that G is not super-λ''. Then there exists a λ''-fragment X of G such that $|\overline{X}| \geqslant |X| \geqslant 4$. Since G is triangle-free, $G[X]$ is also triangle-free and $|E(G[X])| \leqslant \frac{|X|^2}{4}$ by Lemma 4.1. It follows that

$$3k+2 \geqslant \lambda''(G) = d_G(X) = (k+2)|X| - 2|E(G[X])| \geqslant (k+2)|X| - \tfrac{1}{2}|X|^2,$$

that is, $(|X|-3)(|X|-(2k+1))+1 \geqslant 0$, which implies, since $|X| \geqslant 4$ and $k \geqslant 3$,

$$|X| \geqslant 2k+1. \tag{4.8}$$

We will deduce a contradiction to (4.7) by proving that

$$\lambda''(G) > 3k+2. \tag{4.9}$$

To the end, for each $i \in I_m$, let

$$\begin{array}{l} V_i = V(G_i), \quad X_i = X \cap V_i, \\ F_i = E_G(X) \cap E(G_i), \quad F_i' = E_G(X) \cap M_{i,i+1(\text{mod } m)}. \end{array}$$

Then $F_i = E_{G_i}(X_i)$. Let

$$J = \{j \in I_m : \ X_j \neq \emptyset\} \text{ and } J' = \{j \in J : \ X_j = V_j\}.$$

Then $|F_j| \geqslant \lambda(G_j) = k$ for any $j \in J \setminus J'$. Thus, if $|J \setminus J'| \geqslant 4$, then

$$\lambda''(G) = |E_G(X)| \geqslant \sum_{i \in J \setminus J'} |F_i| \geqslant 4k > 3k+2 \quad \text{for } k \geqslant 3,$$

and so (4.9) follows. We assume $|J \setminus J'| \leqslant 3$ below. There are two cases.

Case 1. $|J| \leqslant m-1$.

Subcase 1.1. $J' \neq \emptyset$.

Let $\ell \in I_m \setminus J$ and $j \in J'$. Then $X_\ell = \emptyset$ and $X_j = V_j$. Since $j \neq \ell$, without loss of generality, assume $\ell < j$ and let $s = j - \ell$. By the structure of G, there exist exactly n disjoint paths of length s between V_j and V_ℓ passing through G_{j-1} (maybe $j-1=\ell$), and n disjoint paths of length $m-s$ between V_j and V_ℓ passing through $G_{j+1(\text{mod } m)}$ (maybe $\ell = j+1(\text{mod } m)$). Each of these paths has at least one edge that is in $E_G(X)$. Since $n > \lceil \frac{3k+2}{2} \rceil$, we have that

$$\lambda''(G) = |E_G(X)| \geqslant 2n > 3k+2,$$

and so (4.9) follows.

Subcase 1.2. $J' = \emptyset$. In this subcase, $|J| \leqslant 3$.

If $|J| = 1$, say $X_1 = X$, since $E_G(X) = F_1 \cup F_0' \cup F_1'$ and $|F_0'| = |F_1'| = |X_1| = |X|$, then $|E_G(X)| \geqslant 2|X| + |F_1|$. Combining this fact with (4.8), we have that

$$\lambda''(G) = |E_G(X)| \geqslant 2|X| + |F_1| \geqslant 2(2k+1) + k > 3k+2,$$

and so (4.9) follows.

If $|J|=2$, say $J=\{p,q\}$, then $|p-q|=1$ since $G[X]$ is connected, say $q=p+1$. $F_p\cup F_{p+1}\cup F'_{p-1}\cup F'_{p+1}\subseteq E_G(X)$. Since $|F_p|\geqslant k$, $|F_{p+1}|\geqslant k$ and $|F'_{p-1}\cup F'_{p+1}|=|X|$. Combining these facts with (4.8), we have that

$$\lambda''(G)=|E_G(X)|\geqslant |F_p|+|F_{p+1}|+|X|\geqslant 2k+(2k+1)>3k+2,$$

and so (4.9) follows.

If $|J|=3$, without loss of generality, assume $J=\{1,2,3\}$ since $G[X]$ is connected, then $F'_0\neq\emptyset$ and $F'_3\neq\emptyset$ since $m\geqslant 4$. If $|F'_0|=|F'_3|=1$, then $|X_1|=|X_3|=1$. Since $|X|\geqslant 4$, if $|X_1|=|X_3|=1$, then $|X_2|\geqslant 2$, and so $|F'_1|\geqslant 1$ and $|F'_2|\geqslant 1$. Thus, it is always true that $|F'_0|+|F'_1|+|F'_2|+|F'_3|>2$. It follows that

$$\lambda''(G)=|E_G(X)|\geqslant \sum_{j=1}^{3}|F_j|+\sum_{j=0}^{3}|F'_j|>3k+2,$$

and so (4.9) follows.

Case 2. $|J|=m$.

In this case, $J'\neq\emptyset$ since $m\geqslant 4$ and $|J\setminus J'|\leqslant 3$. If $|J\setminus J'|\leqslant 2$, then $|J'|\geqslant 2$, and

$$|\overline{X}|=\sum_{j\in J\setminus J'}|V(G_j-X_j)|\leqslant 2(n-1)<2n\leqslant\sum_{j\in J'}|V_j|<|X|,$$

a contradiction to $|\overline{X}|\geqslant |X|$. Therefore, $|J\setminus J'|=3$. Since $G[X]$ is connected, without loss of generality, let $J\setminus J'=\{1,2,3\}$. Then $F'_0\neq\emptyset$ and $F'_3\neq\emptyset$. Since $|\overline{X}|\geqslant|X|$, there exists at least two $i\in J\setminus J'$ such that $|X_i|\leqslant\frac{n}{2}$. Thus, at least one of $|V_1\setminus X_1|$ and $|V_3\setminus X_3|$ is not less than $\frac{n}{2}$, that is, either $|F'_0|\geqslant\frac{n}{2}$ or $|F'_3|\geqslant\frac{n}{2}$. It follows that

$$\lambda''(G)=|E_G(X)|\geqslant\sum_{j=1}^{3}|F_j|+|F'_0|+|F'_3|>3k+2,$$

and so (4.9) follows.

Under the hypothesis that G is not super-λ'', we deduce a contradiction to (4.7). Thus, G is super-λ''. By Theorem 3.8 (i), $\rho'(G)=k+1$, and so the theorem follows. ∎

As applications of Theorem 4.7, we consider two families well-known transitive networks.

Let $G(n,d)$ denote a graph which has the vertex-set $V=\{0,1,\ldots,n-1\}$, and two vertices u and v are adjacent if and only if $|u-v|=d^i(\text{mod } n)$ for any i with $0\leqslant i\leqslant\lceil\log_d n\rceil-1$. Clearly, $G(d^m,d)$ is a circulant graph, which is δ-regular and δ-connected, where $\delta=2m-1$ if $d=2$ and $\delta=2m$ if $d\neq 2$. For circulant graphs with order between d^m and d^{m+1}, that is, $G(cd^m,d)$ with $1<c<d$, $\delta=2m+1$ if $c=2$ and $\delta=2m+2$ if $c>2$, moreover, Park and Chwa [15] showed that $G(cd^m,d)$ can be recursively constructed, that is, $G(cd^m,d)=G(G_0,G_1,\ldots,G_{d-1};\mathscr{M})$, where G_i is isomorphic to $G(cd^{m-1},d)$ for each $i=0,1,\ldots,d-1$, and so $G(cd^m,d)$ is called the *recursive circulant graph*, which is δ-regular and δ-connected. In particular, the graph $G(2^m,4)$ is $2m$-regular $2m$-connected, has the same number of vertices and edges as a hypercube Q_m. However, $G(2^m,4)$ with $m\geqslant 3$ is not isomorphic to Q_m

since $G(2^m, 4)$ has an odd cycle of length larger than 3. Compared with Q_m, $G(2^m, 4)$ achieves noticeable improvements in diameter ($\lceil \frac{3m-1}{4} \rceil$). Thus, the recursive circulant graphs have attracted much research interest in recent ten years (see Park [14, 15] and references therein). Since, when $c \geqslant 3$, $G(cd^0, d)$ is isomorphic to a cycle of length c, $G(cd^m, d)$ contains triangles if $c = 3$.

The *n-dimensional undirected toroidal mesh*, denoted by $C(d_1, \ldots, d_n)$, is defined as the cartesian products $C_{d_1} \times C_{d_2} \times \cdots \times C_{d_n}$, where C_{d_i} is a cycle of length d_i ($\geqslant 3$) for each $i = 1, 2, \ldots, n$ and $n \geqslant 2$. It is known that $C(d_1, \ldots, d_n)$ is a $2n$-regular $2n$-edge-connected transitive graph with girth $g = \min\{4, d_i, 1 \leqslant i \leqslant n\}$. Thus, if $d_i \geqslant 4$ for each $i = 1, 2, \ldots, n$, then $C(d_1, \ldots, d_n)$ is triangle-free. $C(d_1, \ldots, d_n)$ can be expressed as $G(G_0, G_1, \ldots, G_{d_1-1}; \mathscr{M})$, where G_i is isomorphic to $C_{d_2} \times \cdots \times C_{d_n}$ for each $i = 0, 1, \ldots, d_1 - 1$.

Since the two families of networks are transitive, by Corollary 3.11, we can determine the exact values of ρ' when the girth $g \geqslant 5$. By Theorem 4.7, we have the following two stronger results immediately.

Corollary 4.8 *Let c, d, m be three positive integers with $1 < c < d$, $c \neq 3$, $d \geqslant 4$, $m \geqslant 2$. Then $G = G(cd^m, d)$ is super-λ'', and $\rho'(G) = 2m$ if $c = 2$ and $\rho'(G) = 2m+1$ if $c \geqslant 4$.*

Corollary 4.9 *If $n \geqslant 3$ and $d_i \geqslant 4$ for $1 \leqslant i \leqslant n$, then $G = C(d_1, \ldots, d_n)$ is super-λ'' and $\rho'(G) = 2n - 1$.*

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